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CENTRAL FOURIER-STIELTJES TRANSFORMS WITH AN ISOLATED VALUE

BY

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ABSTRACT. Let μ be a central Borel measure on a compact, connected group G. If 0 is isolated in the range of $\hat{\mu}$, then there exists a closed, normal subgroup H of G such that $\pi_H\mu$, the restriction of μ to the cosets of H, is the convolution of an invertible measure with a nonzero idempotent measure. This result extends I. Glicksberg's result for LCA groups. An example is given which shows that this result is false in general for disconnected groups.

1. Introduction. In this paper we extend to compact groups a result obtained by I. Glicksberg [2] for LCA groups. Let μ be a bounded Borel measure on a LCA group G. Glicksberg proved that if 0 is an isolated value in the range of $\hat{\mu}$, then there exists a compact subgroup H of G such that $\pi_H\mu$, the restriction of μ to the cosets of H, is the convolution of an invertible measure with a nonzero idempotent measure. We shall prove this result for central measures on compact, connected groups, concluding additionally that H is a closed normal subgroup. A counterexample to the disconnected case will also be given.

Our result generalizes D. Rider's characterization [10] of the central idempotent measures on compact groups. We will borrow rather extensively from Rider's paper and will have need to quote some results from it. Many of our lemmas and theorems are extensions of his results.

2. Preliminaries. For G a compact group, $\Gamma = \Gamma(G)$ will denote the equivalence classes of irreducible unitary representations of G. Corresponding to each $\alpha \in \Gamma$, we denote by χ_α the character of the class α, by $d(\alpha)$ its degree and by Ψ_α the function $\chi_\alpha/d(\alpha)$. The Fourier-Stieltjes transform of a measure $\mu \in M^Z(G)$, the center of $M(G)$, is defined by

$$\hat{\mu}(\alpha) = \int \overline{\Psi}_\alpha \, d\mu \qquad (\alpha \in \Gamma).$$

We shall denote by $E(\mu)$ the set $\{\alpha \in \Gamma: \hat{\mu}(\alpha) \neq 0\}$, i.e., $E(\mu)$ is the support of $\hat{\mu}$.

Let H be a closed, normal subgroup of G. A measure μ is said to be H-canonical if

$$\mu = \Big(\sum_\alpha c_\alpha d(\alpha)\chi_\alpha\Big) m_H,$$

where the sum is finite, m_H denotes the Haar measure of H, and $1/c_\alpha = \int|\chi_\alpha|^2 dm_H$. If μ is H-canonical, then μ is idempotent and $E(\mu)$ consists of a finite union of

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hypercosets of H. For a discussion of canonical measures and the hypercoset structure of a compact group, see Rider [9], [10].

For H a normal Borel subgroup of G, we define $\pi_H\mu \in M(G)$ by

$$(\pi_H\mu)(E) = \sum_x \mu(E \cap xH) \qquad (\mu \in M(G)),$$

the sum being over distinct coset representatives of H. An argument identical to that in Rudin [11, p. 63] shows that the mapping $\mu \to \pi_H\mu$ is an algebra homomorphism of the convolution algebra $M(G)$. An easy computation shows that π_H maps $M^Z(G)$ into itself.

Finally, we define $I(G)$ by $I(G) = \{\mu \in M^Z(G)\colon \mu \neq 0$ and 0 is isolated in the range of $\hat{\mu}\}$. Thus, if $\mu \in I(G)$, then there exists an $\varepsilon > 0$ such that if $\alpha \in \Gamma$, then $|\hat{\mu}(\alpha)| \geqslant \varepsilon$ or $\hat{\mu}(\alpha) = 0$.

We can now state our main theorem.

THEOREM 1. *Let G be a compact, connected group and let $\mu \in I(G)$. There exists a closed, normal subgroup H of G such that $\pi_H\mu = \nu * \eta$, where ν is an invertible central measure and η is a nonzero H-canonical measure.*

The proof will rely on a structure theorem for compact, connected groups. A theorem of A. Weil [12, p. 91] states that every compact, connected group is a factor group of a group of the form $A \times \prod_I G_\alpha$, where A is abelian, each G_α is a compact, connected, simple Lie group and I is some index set. We first prove Theorem 1 for I finite, then for I countable and finally for I of arbitrary cardinality. We defer until later the extension to factor groups.

3. Finite products. In this section we will prove Theorem 1 for groups of the form $A \times \prod_1^n G_i$, where A is abelian, and each G_i is a compact, connected, simple Lie group. We first define some properties that are satisfied by the groups G_i.

DEFINITION. A group G is said to satisfy Condition I if for all $x \notin Z(G)$, the center of G,

$$\Psi_\alpha(x) \to 0 \quad \text{as } d(\alpha) \to \infty.$$

A group G is said to satisfy Condition II if for each positive integer t there are finitely many $\beta_i \in \Gamma$, each β_i of degree t, such that if $\alpha \in \Gamma$ and $d(\alpha) = t$, then $\alpha = \gamma\beta_i$ for some β_i and $\gamma \in \Gamma$ with $d(\gamma) = 1$.

These conditions were introduced by Rider in [9], where he showed that they were satisfied by the unitary groups. Using results of Ragozin [8], Rider [10] proved that all compact, connected, simple Lie groups satisfy these conditions. This implies that for these groups the number of representations of a given degree is finite since the only linear character on such groups is identically 1. It is obvious that Condition II is also satisfied by abelian groups as well as by finite products of groups each satisfying Condition II.

If G is a compact, connected simple Lie group, then $Z(G)$ is finite. The set $Z^\perp = \{\alpha \in \Gamma\colon \psi_{\alpha|_Z} \equiv 1\}$ is the dual object of the factor group G/Z. This set contains representatives of arbitrarily large degree. For if not, then G/Z would contain an open abelian subgroup (C. C. Moore [7]), contradicting the connectedness of G. Using this fact one can prove that if $\gamma \in \Gamma(Z)$, then there exists a sequence $\{\alpha\} \subset \Gamma(G)$ with $d(\alpha) \to \infty$ such that $\psi_{\alpha|_Z} = \gamma$ for all α (see Rider [9]).

Now consider the collection of subsets $\{yxZy^{-1}\}_{y \in G}$ where $x \notin Z$. We claim that this collection is infinite. If not, then the index $|G/Z\colon C(xZ)/Z| < \infty$ where $C(xZ) = \{yZ\colon xyZ = yxZ\}$, the centralizer of xZ in the group G/Z. Since G/Z is connected, we conclude that $C(xZ) = G$, i.e., $xZ \in Z(G/Z)$. Now choose $\{\alpha\} \subset Z^{\perp}$ with $d(\alpha) \to \infty$. By Condition I, $|\psi_\alpha(x)| \to 0$ as $d(\alpha) \to \infty$, but $|\psi_\alpha(x)| = |\psi_\alpha(xZ)| = 1$ since $xZ \in Z(G/Z)$, a contradiction. Now if $\mu \in M^Z(G)$, then $|\mu|(yxZy^{-1}) = |\mu|(xZ)$. Since μ is bounded, it follows that $|\mu|(xZ) = 0$. Hence, if H is a subgroup of Z, then we have the equality

$$\pi_H \mu = \pi_H(\mu_{|Z}). \tag{1}$$

DEFINITION. A measure $\mu \in M^Z(G)$ is of bounded representation type (b.r.t.) if there exists a positive integer M such that $\hat{\mu}(\alpha) = 0$ if $d(\alpha) > M$.

The idea of the proof is to show that either $\mu_{|Z} \neq 0$, or else μ is of b.r.t. Throughout the remainder of this section, A will denote an abelian group and G_i a compact, connected, simple Lie group.

LEMMA 2. *If $G = \prod_1^n G_i$ and $\mu \in I(G)$, then there exists a closed normal subgroup H of G such that if $\sigma = \mu_{|H}$, then $\sigma \neq 0$, σ is of b.r.t. and $\pi_K \sigma = \pi_K \mu$ for any subgroup K of H.*

PROOF. Assume μ is not of b.r.t. Then there exists a sequence $\{\alpha\} \subset E(\mu)$ with $d(\alpha) \to \infty$, where $\alpha = \alpha_1 \cdots \alpha_n$, $\alpha_i \in \Gamma(G_i)$. Hence, for some i, $d(\alpha_i) \to \infty$. Assume $i = 1$. Let $H_1 = Z(G_1) \times G_2 \times \cdots \times G_n$ and let $\sigma_1 = \mu_{|H_1}$. We claim that $\sigma_1 \neq 0$ and $\sigma_1 \in I(H_1)$. Now

$$\int \overline{\Psi}_\alpha \, d\sigma_1 = \int_G \overline{\Psi}_\alpha \, d\mu - \int_{G-H_1} \overline{\Psi}_\alpha \, d\mu. \tag{2}$$

Since G_1 satisfies Condition I, the Lebesgue Dominated Convergence Theorem implies that the last integral in (2) converges to 0. Since $\alpha \in E(\mu)$ and $\mu \in I(G)$, it follows that $\sigma_1 \neq 0$.

Now let $\gamma \in \Gamma(Z_1)$ and $\beta \in \Gamma(\prod_2^n G_i)$. Choose a sequence $\{\lambda\} \subset \Gamma(G_1)$ with $d(\lambda) \to \infty$ such that $\Psi_{\lambda|Z_1} = \gamma$. Then

$$\int_{H_1} \overline{\lambda \Psi_\beta} \, d\sigma_1 = \int_G \overline{\Psi}_\lambda \overline{\Psi}_\beta \, d\mu - \int_{G-H_1} \overline{\Psi}_\lambda \overline{\Psi}_\beta \, d\mu. \tag{3}$$

As before, the last integral in (3) converges to 0 as $d(\lambda) \to \infty$. Since $\mu \in I(G)$, it follows that $\sigma_1 \in I(H_1)$.

If σ_1 is not of b.r.t., then we can repeat the above process. Thus we obtain a subgroup $H = \prod_1^n H_i$, where $H_i = Z(G_i)$ or G_i, and a measure $\sigma \in I(H)$ of b.r.t. If K is a normal subgroup of H, then K is a normal subgroup of G, and the argument that was used to prove (1) can be easily generalized to show that $\pi_K \mu = \pi_K \sigma$.

Lemma 2 allows us to restrict ourselves to measures of b.r.t. We shall complete the section by showing that such measures can be written as a finite sum of measures (with character coefficients) each of which can be viewed as a measure on an abelian factor group for which Glicksberg's theorem holds.

Let H be a closed, normal subgroup of G and let π be the natural projection of G onto G/H. If $\mu \in M(G)$, we define the projection μ^* of μ onto G/H by the equation

$$\int_{G/H} f\, d\mu^* = \int_G f(\pi(x))\, d\mu \qquad (f \in C(G/H)).$$

The transform of μ agrees with the transform of μ^* on $H^\perp$. Thus, if $\hat{\mu}$ vanishes off $H^\perp$, then μ can be identified with a measure on G/H. This will be the case if μ satisfies the equality $\mu = \mu * m_H$. Also, if ν is a measure on G/H, then there is a measure μ on G such that $\mu^* = \nu$ and $\mu = \mu * m_H$.

LEMMA 3. *Let H be a closed normal subgroup of G and let K be a subgroup of G such that $H \subseteq K \subseteq G$. If $\mu \in M(G)$, then $(\pi_K \mu)^* = \pi_{K/H}\mu^*$.*

PROOF.We shall first show that if μ is supported on xK, then μ^* is supported on $\pi(xK)$. Let U' be an open set in G/H such that $\overline{U'} \cap \pi(xK) = \varnothing$, and let f' be a continuous function supported on U'. If we define $U = \pi^{-1}(U')$ and $f(x) = f'(\pi(x))$, then U is open in G, $\overline{U} \cap xK = \varnothing$, and f is a continuous function supported on U. Thus $\int_{G/H} f' d\mu^* = 0$, which implies that $|\mu^*|(U') = 0$ and hence that μ^* is supported on $\pi(xK)$.

Next, we show that if μ vanishes on Borel subsets of the coset xK, then μ^* vanishes on Borel subsets of the coset $\pi(xK)$. Let $\varepsilon > 0$. By the regularity of μ there is an open set $U \supset xK$ such that $|\mu|(U) < \varepsilon$. Let $U' = \pi(U)$ and let f' be a continuous function supported on U' such that $\|f'\|_\infty \leq 1$. If $f(x) = f'(\pi(x))$, then

$$\left|\int_{G/H} f'\, d\mu^*\right| = \left|\int_G f\, d\mu\right| < \varepsilon.$$

Thus, $|\mu^*|(U') < \varepsilon$, and hence $|\mu^*|(\pi(xK)) = 0$ by regularity.

Now if $\mu \in M(G)$, then $\mu = \pi_K\mu + \sigma$, where σ vanishes on Borel subsets of cosets of K, and $\mu^* = \pi_{K/H}\mu^* + \sigma_0$, where σ_0 vanishes on the Borel subsets of cosets of K/H. Thus $\pi_{K/H}\mu^* + \sigma_0 = \mu^* = (\pi_K\mu)^* + \sigma^*$. But $(\pi_K\mu)^*$ is supported on cosets of K/H and σ^* vanishes on cosets of K/H, and thus $\pi_{K/H}\mu^* = (\pi_K\mu)^*$ by the uniqueness of the decomposition.

LEMMA 4. *If $G = A \times \prod_1^n G_i$ and $\mu \in I(G)$ is of b.r.t., then $\mu = \sum_1^n d(\beta_i)\chi_{\beta_i}\lambda_i$, where each λ_i satisfies $\lambda_i = \lambda_i * m_{G'}$ with $G' = \prod_1^n G_i$.*

PROOF. Let $\beta_i \in \Gamma(G')$ and put $\theta_i = d(\beta_i)\chi_{\beta_i} m_{G'}$. Then θ_i is a central idempotent and $E(\theta_i) = \beta_i\Gamma(A)$. Since μ is of b.r.t. and G satisfies Condition II, μ is a finite sum of measures of the form $\mu * \theta_i$. Put $\lambda_i = \psi_{\beta_i}\mu * m_{G'}$. An elementary calculation (see Rider [9]) shows that $\mu * \theta_i = d(\beta_i)\chi_{\beta_i}\lambda_i$.

It is clear in the proof of Lemma 4 that 0 is an isolated value in the range of each $\hat{\lambda}_i$. Since $\lambda_i = \lambda_i * m_{G'}$, we can view λ_i as an element of $I(A)$ and apply Glicksberg's result to λ_i. However, the λ_i may yield different H_i. The next lemma deals with this problem.

LEMMA 5. *Let* $\lambda_i \in I(A)$ *for* $1 \leqslant i \leqslant n$, *then there exists a compact subgroup* H *of* A *such that, for all* i, $\pi_H\lambda_i = \nu_i * \eta_i$, *where* ν_i *is invertible and* η_i *is* H*-canonical, and at least one* $\eta_i \neq 0$.

PROOF. We will assume $n = 2$, the general case following easily by induction. We apply Glicksberg's result to λ_1 and λ_2, so that there exist compact subgroups H and K such that $\pi_H\lambda_1 = \nu_1 * \eta_1$ and $\pi_K\lambda_2 = \nu_2 * \eta_2$, where η_1 is H-canonical and η_2 is K-canonical. The proof is now divided into three cases.

Case I. $H \cap K$ is open in H and in K.

In this case, $\pi_H = \pi_{HK} = \pi_K$, and it is easily seen that we can replace both H and K by HK.

Case II. $H \cap K$ is open in H but not in K.

In this case, $\pi_H = \pi_{H\cap K} = \pi_{HK}$. The last equality is true for any two subgroups of A. Thus

$$\pi_H\lambda_2 = \pi_H\pi_K\lambda_2 = \pi_H(\nu_2 * \eta_2) = \pi_H\nu_2 * \pi_H\eta_2 = 0$$

since $\pi_H m_K = 0$.

The case where $H \cap K$ is open in K but not in H is handled identically.

Case III. $H \cap K$ is open neither in H nor in K.

Clearly, we can assume $\pi_K\lambda_1 \neq 0$. A theorem of Glicksberg and Wik [3] states that the range of $(\pi_K\lambda_1)\hat{}$ is contained in the closure of the range of $\hat{\lambda}_1$. Thus $\pi_K\lambda_1 \in I(A)$, so that there exists a subgroup L of A such that $0 \neq \pi_L\pi_K\lambda_1 = \nu * \eta$, where η is L-canonical. Since $\pi_K\eta \neq 0$, it follows that $L \cap K$ is open in L. Thus $\pi_L\lambda_1 = \pi_L\pi_K\lambda_1 = \nu * \eta$. If $\pi_L\lambda_2 \neq 0$, then $0 \neq \pi_L\lambda_2 = \pi_L\pi_K\lambda_2 = \pi_L(\nu_2 * \eta_2)$. Hence $\pi_L\eta_2 \neq 0$ and $L \cap K$ is open in K. The problem has now been reduced to Case I.

THEOREM 6. *If* $G = A \times \prod_1^n G_i$ *and* $\mu \in I(G)$ *is of b.r.t., then there exists a closed, normal subgroup* H *of* G *such that* $0 \neq \pi_H\mu = \nu * \eta$, *where* ν *is invertible and* η *is* H*-canonical.*

PROOF. By Lemma 4, $\mu = \sum_1^m d(\beta_i)\chi_{\beta_i}\lambda_i$, where $\lambda_i = \psi_{\beta_i}\mu * m_{G'}$ for all i. The projection λ_i^* of λ_i onto G/G' is an element of $I(G/G')$, and so by Lemma 5 (since $G/G' \simeq A$) there exists a closed, normal subgroup H/G' of G/G' such that $\pi_{H/G'}\lambda_i^* = \nu_i * \eta_i$, where $\eta_i \ll m_{H/G'}$ and $\nu_i \in M^{-1}(G/G')$ for all i. Since at least one $\eta_i \neq 0$, we can assume $\eta_i \neq 0$ for all i. Identify ν_i and η_i with measures on G, so that $\nu_i * m_{G'} = \nu_i$ and $\eta_i * m_{G'} = \eta_i$. Thus $\eta_i \ll m_H$ for all i and there exist measures ν_i' on G such that $\nu_i * \nu_i' = m_{G'}$. (Note that $m_{G'}$ is identified with the identity measure in $M(G/G')$.)

Now by Lemma 3, $(\pi_H\lambda_i)^* = \pi_{H/G'}\lambda_i^*$. Thus, $\pi_H\lambda_i * m_{G'} = \nu_i * \eta_i$. This implies that $\pi_H\lambda_i \neq 0$, and since $\lambda_i * m_{G'} = \lambda_i$, we have

$$0 \neq \pi_H\lambda_i = \pi_H\lambda_i * \pi_H m_{G'} = \pi_H\lambda_i * m_{G'} = \nu_i * \eta_i.$$

Therefore,

$$\pi_H\mu = \sum_1^m d(\beta_i)\chi_{\beta_i}(\nu_i * \eta_i).$$

Put $\nu = \Sigma_1^m d(\beta_i)\chi_{\beta_i}\nu_i$ and $\eta = \Sigma_1^m d(\beta_i)\chi_{\beta_i}\eta_i$. If $\gamma \in \Gamma(A)$, then

$$\begin{aligned}(\pi_H\mu)\hat{}(\gamma\beta_i) &= \big(d(\beta_i)\chi_{\beta_i}(\nu_i * \eta_i)\big)\hat{}(\gamma\beta_i)\\ &= \int \bar{\gamma}|\chi_{\beta_i}|^2 d(\nu_i * \eta_i)\\ &= \int \bar{\gamma} d(\nu_i * \eta_i) = \hat{\nu}_i(\gamma)\hat{\eta}_i(\gamma).\end{aligned}$$

On the other hand,

$$\begin{aligned}(\nu * \eta)\hat{}(\gamma\beta_i) &= \big(d(\beta_i)\chi_{\beta_i}\nu_i\big)\hat{}(\gamma\beta_i') \cdot \big(d(\beta_i)\chi_{\beta_i}\eta_i\big)\hat{}(\gamma\beta_i)\\ &= \int \bar{\gamma}|\chi_{\beta_i}|^2\, d\nu_i \cdot \int \bar{\gamma}|\chi_{\beta_i}|^2\, d\eta_i = \hat{\nu}_i(\gamma)\hat{\eta}_i(\gamma).\end{aligned}$$

Therefore, $\pi_H\mu = \nu * \eta$, and if we now let $\eta_i' = d(\beta_i)\chi_{\beta_i}\eta_i$ and $\nu_0 = \nu + \Pi_1^m(\delta_0 - \eta_i')$, then $\pi_H\mu = \nu_0 * \eta$. It is clear that $\eta \ll m_H$ and the inverse of ν_0 is given by $\nu_0^{-1} = \Sigma_1^m d(\beta_i)\chi_{\beta_i}\nu_i' + \Pi_1^m(\delta_0 - \eta_i')$.

COROLLARY 7. *If $G = \Pi_1^n G_i$ and $\mu \in I(G)$ is of b.r.t. then μ is a trigonometric polynomial.*

4. Countable products. In this section we consider the case when $G = A \times \Pi_1^\infty G_i$, where A and G_i are as in the preceding section. If $\mu \in M^z(G)$, then we shall say that μ is of bounded representation type in each coordinate if there exist positive integers M_i, $1 \leqslant i \leqslant \infty$, such that if $\alpha = \alpha_1 \cdots \alpha_n \in E(\mu)$ and $d(\alpha_i) > M_i$ for some i, then $\hat{\mu}(\alpha) = 0$. We shall first reduce the problem to measures which are of b.r.t. in each coodinate. We then shall prove that if μ is such a measure and $\mu \in I(G)$, then there exists an N such that $\mu = \mu * m_{H'}$ where $H = \Pi_{N+1}^\infty G_i$. Hence we can treat μ as a measure on $A \times \Pi_1^N G_i$ and apply the results of the previous section.

We note that by replacing μ by $\mu * \tilde{\mu}$, where $\tilde{\mu}(E) = \overline{\mu(E^{-1})}$, we can assume that $\hat{\mu}$ is positive. It is also clear that we can assume $\hat{\mu}(\alpha) > 1$ for all $\alpha \in E(\mu)$. If μ is not of b.r.t. in the ith coodinate, put $H_i = Z_i$; otherwise put $H_i = G_i$. Let $K_n = A \times H_1 \times \cdots \times H_n \times G_{n+1} \times \cdots$ and $\sigma_n = \mu_{|K_n}$. The argument given in Lemma 2 shows that $\sigma_n \neq 0$ and that $\hat{\sigma}_n(\alpha) \geqslant 1$ for all $\alpha \in E(\sigma_n)$. Thus $\|\sigma_n\| \geqslant 1$, i.e., $|\mu|(K_n) \geqslant 1$. Put $K = \bigcap_{n=1}^\infty K_n = A \times \Pi_1^\infty H_i$ and put $\sigma = \mu_{|K}$. Since $|\mu|(K) = \lim_{n\to\infty}|\mu|(K_n) \geqslant 1$, it follows that $\sigma \neq 0$.

We claim that $\sigma \in I(K)$. Let $\alpha = \gamma\alpha_1 \cdots \alpha_n \in \Gamma(K)$. If $H_i = Z_i$, choose $\{\lambda_i^j\}_{j=1}^\infty \subset \Gamma(G_i)$ such that $d(\lambda_i^j) \to \infty$ as $j \to \infty$ and such that $\Psi_{\lambda_i^j|Z_i} = \alpha_i$ if $i \leqslant n$ and $\Psi_{\lambda_i^j|Z_i} = 1$ if $i > n$. If $H_i = G_i$, choose $\{\lambda_i^j\} \subset \Gamma(G)$ such that $\lambda_i^j = \alpha_i$ for all j. Now put $\beta_m = \lambda_1^m \cdots \lambda_m^m$ for $m \geqslant n$. Now if $x \notin K$ then $x_i \notin Z_i$ for some i for which $H_i = Z_i$, so that $\lim_{m\to\infty}\Psi_{\beta_m}(x) = 0$. Since $\Psi_{\beta_m|K} = \Psi_\alpha$, we have the equality

$$\int \overline{\Psi}_\alpha d\sigma = \int \overline{\Psi}_{\beta_m} d\sigma = \int \overline{\Psi}_{\beta_m} d\mu - \int \overline{\Psi}_{\beta_m} d\mu_{|G-K}.$$

The last integral above converges to 0 by the Lebesgue Dominated Convergence Theorem. Since $\mu \in I(G)$, it follows that $\sigma \in I(K)$, and since $\pi_K\mu = \pi_K\sigma$, we may assume μ is of b.r.t. in each coodinate.

Our next lemma deals with norms of polynomials and generalizes a result of Rider [**10**, p. 466]. Our proof is a slight modification of his.

LEMMA 8. *Let G be a compact, connected group and let* $P = \Sigma c_\alpha d(\alpha)\chi_\alpha$ *be a central polynomial on G with* $c_\alpha \geqslant 1$ *for all* α. *If* $\|P\|_1 < 1 + 1/300$, *then* $P = c\gamma$, *where* γ *is a linear character and* $1 \leqslant c \leqslant 1 + 1/300$.

PROOF. Write $|P|^2 = P\bar{P} = \Sigma a_\alpha \chi_\alpha$, $a_\alpha \geqslant 1$, and $|P|^4 = b_\alpha \chi_\alpha$, $b_\alpha \geqslant 1$. Let

$$M = \|P\|_2^2 = \int P\bar{P} = \sum \int c_\alpha^2 d^2(\alpha)\chi_\alpha \bar{\chi}_\alpha$$
$$= \sum c_\alpha^2 d^2(\alpha) \geqslant \sum c_\alpha d^2(\alpha) = \|P\|_\infty.$$

We have the inequalities

$$a_\alpha = \int |P|^2 \bar{\chi}_\alpha \leqslant d(\alpha)\|P\|_2^2 = Md(\alpha)$$

and

$$b_\alpha = \int |P|^4 \bar{\chi}_\alpha \leqslant d(\alpha)\|P\|_\infty^2 \int |P|^2 \leqslant d(\alpha)M^2 \cdot M = d(\alpha)M^3.$$

By Hölder's inequality,

$$\sum a_\alpha b_\alpha = \int |P|^6 \geqslant M^5 \|P\|_1^{-4}.$$

Define

$$A_1(g) = M^{-4} \sum b_\alpha\left(1 - a_\alpha (Md(\alpha))^{-1}\right)\chi_\alpha(g)$$

and

$$A_2(g) = M^{-2} \sum a_\alpha\left(1 - b_\alpha (M^3 d(\alpha))^{-1}\right)\chi_\alpha(g).$$

Then

$$\|A_i\|_\infty = A_i(e) \leqslant 1 - \|P\|_1^{-4} \qquad (i = 1, 2).$$

Thus if $\|P\|_1 \leqslant 1 + 1/300$, then $\|A_i\|_\infty \leqslant 1/60$. It follows that $1/30 \geqslant A_2 - A_1 = |P|^2/M^2 - |P|^4/M^4 \geqslant 0$. Thus either $|P(g)| \leqslant M/5$ or $|P(g)| \geqslant 4M/5$. Suppose $P(e) = \Sigma c_\alpha d^2(\alpha) \leqslant M/5$. Then

$$0 \leqslant \frac{\sum c_\alpha^2 d^2(\alpha)}{5} - \sum c_\alpha d^2(\alpha) = \sum c_\alpha d^2(\alpha)\left(\frac{c_\alpha}{5} - 1\right).$$

Since $c_\alpha \leqslant \|P\|_1 < 5$, $\Sigma c_\alpha d^2(\alpha)(c_\alpha/5 - 1) < 0$, a contradiction. Since G is connected, $P(g) \geqslant 4M/5$ for all $g \in G$. Thus $1 + 1/300 \geqslant \|P\|_1 > 4M/5$, which yields $M < 2$, and the result follows.

LEMMA 9. *Let* $G = \Pi_1^\infty G_i$ *and let* $\mu \in I(G)$ *be of b.r.t. in each coordinate. If* $\alpha \in E(\mu)$ *if and only if* $\|(\Psi_\alpha - 1)\mu\| < 1/600$, *then* $\mu = cm_G$ *for some constant c.*

PROOF. Since $|\hat{\mu}(\alpha) - \hat{\mu}(1)| < 1/600$ for all $\alpha \in E(\mu)$, we can assume $1 \leq \mu(\alpha) \leq 1 + 1/600$ for all $\alpha \in E(\mu)$. Let $G^r = \Pi_1^r G_i$ and $G^{r'} = \Pi_{r+1}^{\infty} G_i$, and let $\mu_r = \mu * m_{G^{r'}}$. We shall show that μ_r^*, which we can identify as an element of $I(G^r)$, is of the form $c_r m_{G^r}$ for some constant c_r. It follows that $\mu_r = c_r m_{G^r} * m_{G^{r'}}$. The result follows since $\mu_r \to \mu$, w^*, and $c_r m_{G^r} * m_{G^{r'}} \to c m_G$, w^*, for some constant c.

Since μ is of b.r.t. in each coodinate, it follows that μ_r^* is of b.r.t. By Corollary 7, μ_r^* is a polynomial $P = \Sigma c_\alpha d(\alpha)\chi_\alpha$ on G_r. If $\alpha \in E(\mu_r^*)$, then $\|(\overline{\Psi}_\alpha - 1)\mu_r^*\| < 1/600$. Now

$$\begin{aligned}\int\left|\left(\frac{P}{\|P\|_\infty} - 1\right)P\right| dm_{G_r} &= \int \frac{|(P - P(e))P|}{\|P\|_\infty}\, dm_{G_r}\\ &= \frac{1}{\|P\|_\infty}\int\left|\left(\sum c_\alpha d(\alpha)\chi_\alpha - \sum c_\alpha d^2(\alpha)\right)P\right| dm_{G_r}\\ &= \frac{1}{\|P\|_\infty}\int\left|\left(\sum c_\alpha d^2(\alpha)\left(\frac{\chi_\alpha}{d(\alpha)} - 1\right)\right)P\right| dm_{G_r}\\ &\leq 1/600.\end{aligned}$$

On the other hand

$$\int\left|\left(\frac{P}{P(e)} - 1\right)P\right| \geq \int |P| - \frac{\int |P|^2}{P(e)} = \|P\|_1 - \frac{\sum c_\alpha^2 d^2(\alpha)}{\sum c_\alpha d^2(\alpha)}.$$

Therefore,

$$\begin{aligned}\|P\|_1 &\leq \frac{1}{600} + \frac{\sum c_\alpha^2 d^2(\alpha)}{\sum c_\alpha d^2(\alpha)}\\ &\leq \frac{1}{600} + \frac{\sum (1 + 1/600) c_\alpha d^2(\alpha)}{\sum c_\alpha d^2(\alpha)} = 1 + \frac{1}{300}.\end{aligned}$$

By Lemma 8, $P = c_r m_{G_r}$ for some constant c_r.

Our next lemma is of a very technical nature. It generalizes Rider's Lemma 6.3 of **[10]**. First, we need to introduce the concept of an irreducible sequence.

DEFINITION. A sequence $\{\alpha_i\} \subset \Gamma(G)$ is said to be *irreducible* if, for fixed $\beta \in \Gamma$, $\alpha_i \otimes \beta$ is eventually irreducible. Let $G = \Pi_1^\infty G_i$ and let $\{\alpha_i\} \subset \Gamma(G)$, with $\alpha_i = \alpha_i^{(1)}\alpha_i^{(2)} \cdots \alpha_i^{(n)}$ where $\alpha_i^{(j)} \in \Gamma(G_j)$. The sequence $\{\alpha_i\}$ is called an *F-sequence* if for each positive integer N there exists a positive integer N' such that if $i > N'$ then $d(\alpha_i^{(j)}) = 1$ for all $j \leq N$. An *F-sequence* is an irreducible sequence.

LEMMA 10. *Let $G = A \times \Pi_1^\infty G_i$, let n be a positive integer, and let $M < \infty$. There exists a $\delta > 0$ such that if $\mu \in I(G)$ satisfies the following*:

(a) $\|\mu\| \leq M$.

(b) *There exists a set B of Borel homomorphisms f into the unit circle. Each $f \in B$ is defined on a normal Borel subgroup T_f of G with $|\mu|(T_f^c) = 0$.*

(c) *The function* $f_1 = 1 \in B$, *and if* $f \in B$ *then there is an F-sequence* $\{\alpha\} \subset E(\mu)$ *with* $\overline{\Psi}_\alpha \mu \to f\mu$ *in norm, and for every F-sequence* $\{\alpha\} \subset E(u)$, $\overline{\Psi}_\alpha \mu \to f\mu$ *in norm (for a subsequence of* $\{\alpha\}$*) for some* $f \in B$.

(d) *The set B is partitioned into finitely many subsets* $B_1, \ldots, B_n$ *by* $f \sim g$ *if* $E(f\mu) = E(g\mu)$.

(e) $\alpha \in E(\mu)$ *if and only if* $\|(\overline{\Psi}_\alpha - f)\mu\| < \delta$ *for some* $f \in B$, *then there exist* $h_i \in L^\infty(|\mu|)$, $1 \leqslant i \leqslant n$, *such that if* $\sigma = h_1\mu * \cdots * h_n\mu$, *then* $\|(\overline{\Psi}_\alpha - 1)\sigma\| < 1/600$ *if and only if* $\alpha \in E(\sigma)$.

PROOF. We shall first show that if f, g and $h \in B$ with $f \sim g$, then $E(fh\mu) = E(gh\mu)$. Note that we do not claim that $fh \in B$. Since $\|fh\mu\| = \|\mu\|$, we have that $fh\mu \neq 0$. Let $\overline{\Psi}_\beta \mu \to g\mu$ and $\overline{\Psi}_\gamma \mu \to h\mu$. If $\alpha \in E(fh\mu)$, then $0 \neq \int \overline{\Psi}_\alpha fh\, d\mu = \lim_\gamma \int \overline{\Psi}_\alpha \overline{\Psi}_\gamma f\, d\mu$. Thus, for $\gamma > \gamma_0$, $\int \overline{\Psi}_\alpha \overline{\Psi}_\gamma f\, d\mu \neq 0$ and $\alpha \otimes \gamma$ is irreducible. This implies that $\alpha\gamma \in E(f\mu) = E(g\mu)$, so that $0 \neq \int \overline{\Psi}_\alpha \overline{\Psi}_\gamma\, g\, d\mu = \lim_\beta \int \overline{\Psi}_\alpha \overline{\Psi}_\gamma \overline{\Psi}_\beta\, d\mu$ if $\gamma > \gamma_0$. Since $\alpha \otimes \gamma \otimes \beta$ is irreducible for β large enough, $\int \overline{\Psi}_\alpha \overline{\Psi}_\gamma\, g\, d\mu \geqslant 1$. Thus, $\int \overline{\Psi}_\alpha hg\, d\mu = \lim_\gamma \int \Psi_\alpha \Psi_\gamma g\, d\mu \geqslant 1$, i.e., $\alpha \in E(gh\mu)$. Similarly, $E(gh\mu) \subset E(fh\mu)$.

Let $\Lambda = B_1$ be the equivalence class containing $f_1 = 1$. If $f, g \in \Lambda$ then $E(fg\mu) = E(ff_1\mu) = E(f\mu) = E(\mu)$, i.e., Λ is a semigroup under function multiplication. Topologize Λ by defining $\|f - g\|_\Lambda = \|f\mu - g\mu\|_{M(G)}$. If $\{f_i\} \subset \Lambda$, then there exists a $\nu \in M(G)$ such that $f_i\mu \to \nu$, w^* (for some subsequence). By using a diagonal process, one can construct an F-sequence $\{\alpha\} \subset E(\mu)$ such that $\overline{\Psi}_\alpha \mu \to \nu$, w^*. From (c), $\overline{\Psi}_\alpha \mu \to f\mu$ in norm (for a subsequence) for some $f \in B$. Thus $\nu = f\mu$, and from (b), $\|\nu\| = \|\mu\|$. Since $\{f_i\} \subset \Lambda$, it follows that $f \in \Lambda$. Using a theorem of Amemiya and Ito (see Rider [**10**, Lemma 7.5]), $f_i\mu \to \nu$ in norm, and hence Λ is a compact abelian semigroup. It follows [**5**, p. 99] that Λ is a compact group. Hence, there exists a Haar measure m on Λ.

Choose fixed representatives $f_i \in B_i$ with $f_1 \equiv 1$, and define h_i by the vector-valued integral

$$h_i = \int_\Lambda f_i f\, dm(f), \qquad f \in \Lambda.$$

Note that $h_i = f_i \int_\Lambda f\, dm(f) = f_i h_1$. Let $\sigma = h_1\mu * \cdots * h_n\mu$. Then $\sigma \neq 0$ since $1 \in \bigcap E(h_i\mu) = E(\sigma)$. Also, $\sigma \in I(G)$ since each $h_i\mu \in I(G)$.

Let β be a fixed element of $E(\sigma)$. Then $\beta \in E(f_i\mu)$ for all i. If $\overline{\Psi}_\alpha \to f_i$, as in (c), then $\beta \otimes \alpha$ is eventually irreducible and $\beta\alpha \in E(\mu)$. By (e), $\|(\overline{\Psi}_\beta \overline{\Psi}_\alpha - g_j)\mu\| < \delta$ for some $g_j \in B_j$. Since $\overline{\Psi}_\alpha \to f_i$ a.e. $(|\mu|)$, we have that

$$\left\|\left(\overline{\Psi}_\beta - \bar{f}_i g_j\right)\mu\right\| = \left\|\left(\overline{\Psi}_\beta f_i - g_j\right)\mu\right\| \leqslant \delta. \tag{4}$$

Since $f_1 \equiv 1$, there is a g_K such that

$$\left\|\left(\overline{\Psi}_\beta - g_K\right)\mu\right\| < \delta. \tag{5}$$

We will show that $g_K\sigma = \sigma$.

From (4) and (5) it follows that (choosing $\delta < 1/2$)

$$\left\|\left(g_K - \bar{f}_i g_j\right)\mu\right\| = \left\|\left(g_K f_i - g_j\right)\mu\right\| < 1.$$

Thus $1 \in E(g_K f_i \mu)$ and hence $g_K f_i \in B_j$. Since $g_K f_i \sim f_j$, it follows that $E(\bar{f}_j g_K f_i \mu) = E(\bar{f}_j f_j \mu) = E(\mu)$. Thus $h = \bar{f}_j g_K f_i \in \Lambda$ and $g_K f_i = f_j h$. We now have that

$$g_K h_i \mu = g_K \left(\int_\Lambda f_i f \, dm \right) \mu = \left(\int_\Lambda g_K f_i f \, dm \right) \mu$$
$$= \left(\int_\Lambda f_j h f \, dm \right) \mu = \left(\int_\Lambda f_j f \, dm \right) \mu = h_j \mu.$$

Also, if $g_K h_i \mu = g_K h_j \mu$, then $g_K f_i \sim g_K f_j$, which implies that $f_i \sim f_j$. Thus the mapping $h_i \mu \to g_K h_i \mu$ is a permutation of the $\{h_i \mu\}$. Since $\operatorname{supp} \sigma \subset \operatorname{supp} \mu$ and $|\mu|(T^c_{g_K}) = 0$, we have that

$$g_K \sigma = g_K (h_1 \mu * \cdots * h_n \mu) = g_K h_1 \mu * \cdots * g_K h_n \mu = \sigma.$$

Now let $R = \{x \in \operatorname{supp} \mu : |\Psi_\beta(x) - g_K(x)| < \sqrt{\delta}\}$. It follows from (5) that $|\mu|(R^c) \leqslant \sqrt{\delta}$. A result of Rider [**10**, Lemma 5.1] implies that, on $R^n = R \times \cdots \times R$, we have the inequality

$$|\Psi_\beta(x_1 \cdots x_n) - g_K(x_1 \cdots x_n)| \leqslant n^2 \sqrt{\delta}, \qquad x_i \in R \ (1 \leqslant i \leqslant n).$$

Now

$$\left\| (\bar{\Psi}_\beta - 1)\sigma \right\| = \left\| (\bar{\Psi}_\beta - g_K)\sigma \right\|$$
$$\leqslant \int |\bar{\Psi}_\beta(x_1 \cdots x_n) - g_K(x_1 \cdots x_n)| d(|\mu| \times \cdots \times |\mu|)$$
$$\leqslant n^2 \delta^{1/2} M^n + 2nM^{n-1} \delta^{1/2} < 1/600$$

for δ small enough. The second inequality above follows by integrating over R^n and $(R^n)^c$ separately.

The following lemma extends a result of Glicksberg's [**2**] to compact groups. It is a generalization of Helson's translation lemma.

LEMMA 11. *Let G be a compact group and suppose $\Psi_\alpha \mu \to \omega$, w^*, where $\{\Psi_\alpha\} \subset \Gamma$ is an irreducible sequence. If $0 \neq \omega = \omega * \eta$, where η is H-canonical, then $\{\Psi_\alpha\}$ is contained in a finite number of hypercosets of $H^\perp$.*

PROOF. The proof follows Glicksberg exactly. Since $\omega = \omega * \eta$, it follows that $h \to \delta_h * \omega$ is norm-continuous on H. Let $\mu = \rho + \sigma$ be the Lebesgue decomposition of μ with respect to $|\omega|$ where $\rho \ll |\omega|$ and $\sigma \perp |\omega|$. Thus $h \to \delta_h * \rho$ is also norm-continuous on H, and therefore we can approximate ρ in norm by $\int_H \delta_h * \rho \cdot f(h) \, dm_H = \rho * f m_H$, where $f \in L_1^z(H)$. If the conclusion of the lemma is false, then $\Psi_\alpha f m_H \to 0$, w^*, by the Riemann-Lebesgue lemma, so that $\Psi_\alpha(\rho * f m_H) \to 0$, w^*, since $\{\alpha\}$ is an irreducible sequence. Thus $\Psi_\alpha \rho \to 0$, w^*, and $\omega = \lim \Psi_\alpha \sigma$. Now let g be a w^*-cluster point of $\{\Psi_\alpha\}$ in $L^\infty(|\sigma|)$. Then $\omega = g\sigma$ but $\omega \perp \sigma$, a contradiction.

The following lemma can be found in Rider [**10**, p. 471].

LEMMA 12 (RIDER). *Let H be a closed normal subgroup of a compact group G. Suppose $\{\Psi_\alpha\}$ is a sequence in Γ such that $|\Psi_\alpha(x)| \to 1$ a.e. (m_H). Then, for α large enough, $|\Psi_\alpha(x)| \equiv 1$ on H.*

THEOREM 13. *Let* $G = A \times \prod_1^\infty G_i$ *and let* $\mu \in I(G)$ *be of b.r.t. in each coordinate. There exists an integer* N *such that* $\mu = \mu * m_H$, *where* $H = \prod_{N+1}^\infty G_i$.

Thus μ can be considered as a measure on $A \times \prod_1^N G_i$ and we can apply the results of §3. Note that the theorem implies that μ is of b.r.t.

PROOF. As noted at the beginning of this section, we can assume $\hat{\mu}(\alpha) \geqslant 1$ for all $\alpha \in E(\mu)$. If $\|\mu\| = 1$, then μ is an idempotent and hence $\mu = \gamma m_K$, where $K = A_0 \times \prod_1^\infty G_i$ with A_0 a compact subgroup of A and $\gamma \in \Gamma(A_0)$ (see Greenleaf [4, Theorem 2.1.4]). Now assume the theorem is true for all measures of norm less than A and assume $1 \leqslant A \leqslant \|\mu\| < A + 1/100A$.

Assume the theorem is false. Using the fact that μ is of b.r.t. in each coordinate and that each G_i satisfies Condition II, we obtain a sequence $\{\alpha_n\} \subset E(\mu)$ with the following properties.

(a) $\alpha_n = \beta_1 \cdots \beta_n \gamma_n \lambda_n$, where $\beta_i \in \Gamma(G_i)$, $\gamma_n \in \Gamma(A)$ and $\lambda_n \in \Gamma(\prod_{n+1}^\infty G_i)$.

(b) $d(\lambda_n) > 1$ for each n.

Let $P = \{x \in G\colon |\Psi_{\beta_i}(x)| \to 1 \text{ as } i \to \infty\}$. Then P is a normal Borel subgroup of G which contains $A \times \prod_1^m G_i$ for all m. We shall show that $P = G$ and hence, by Lemma 12, that $\Psi_{\beta_i} \equiv 1$ for i large enough. Now

$$\int |\Psi_{\alpha_n}| d|\mu - \pi_P\mu| \to 0 \quad \text{as } n \to \infty$$

by the Lebesgue Dominated Convergence Theorem. Thus we can choose an N such that

$$\int |\Psi_{\alpha_N}| d|\mu - \pi_P\mu| < \frac{1}{4}. \tag{6}$$

Define $K = A \times \prod_1^{N_0} G_i$, where $N_0 > N$ is chosen so that $\alpha_N \in \Gamma(K)$, and put $\lambda = \overline{\Psi}_{\alpha_N}\mu * m_K$. Then $1 \in E(\lambda)$, $\lambda \in I(G)$ and λ is of b.r.t. in each coordinate. If $\alpha \in E(\lambda)$, then

$$\hat{\lambda}(\alpha) = \int \overline{\Psi}_\alpha \, d\lambda = \int \overline{\Psi}_\alpha \overline{\Psi}_{\alpha_N} \, d\pi_P\mu + \int \overline{\Psi}_\alpha \overline{\Psi}_{\alpha_N} \, d(\mu - \pi_P\mu). \tag{7}$$

From (6) and (7) it follows that $|(\pi_P\lambda)\hat{}(\alpha)| \geqslant 3/4$ if $\alpha \in E(\lambda)$. In particular, $E(\lambda) \subseteq E(\pi_P\lambda)$.

Let $B = \{\omega\colon \omega = w^*\text{-lim } \Psi_{t_n}\lambda$, where $\{t_n\}$ is an F-sequence in $E(\lambda)\}$. Since $1 \in E(\lambda)$, $\varnothing \neq B \subseteq I(G)$. Also, $\omega \in B$ implies that $1 \in E(\omega)$. Using diagonal sequences one can show that B is closed in the w^* topology. Since $0 \notin B$, it follows that B contains an element ω_0 of minimal norm.

Suppose $\|\omega_0\| < A$. The induction hypothesis implies that ω_0 is a trigonometric polynomial on G. If $\omega_0 = \lim \overline{\Psi}_{t_n}\lambda$, then Lemma 11 implies that $\Psi_{t_n} \equiv 1$ for n large enough, and hence $\omega_0 = \lambda$. Since $\pi_P\lambda \neq 0$, it follows that P is open in G and thus $P = G$.

Now suppose $A \leqslant \|\omega_0\| < A + 1/100A$. If $\omega_0 = \lim \overline{\Psi}_{t_n}\lambda$, a result of Rider [10, Lemma 7.4] shows that $\|\overline{\Psi}_{t_n}\lambda - \overline{\Psi}_{t_m}\lambda\| < 1/4$ for n, m large enough. Letting $m \to \infty$, we have the inequality $\|\overline{\Psi}_{t_n}\lambda - \omega_0\| \leqslant 1/4$, and hence

$$\|\overline{\Psi}_{t_n}\pi_P\lambda - \pi_P\omega_0\| \leqslant 1/4 \quad \text{for } n \text{ large enough.} \tag{8}$$

From (8) and the fact that $|(\pi_P\lambda)\hat{}(\alpha)| \geq 3/4$ if $\alpha \in E(\lambda)$, it follows that $E(\omega_0) \subseteq E(\pi_P\omega_0)$.

Now let $B_0 = \{\sigma\colon \sigma = w^*\text{-lim } \overline{\Psi}_{t_n}\omega_0$, where $\{t_n\}$ is an F-sequence in $E(\omega_0)\}$. It is not difficult to show that $B_0 \subseteq B$, and since ω_0 was chosen of minimal norm in B, it follows that $\|\sigma\| = \|\omega_0\|$ if $\sigma \in B_0$. By Rider [**10**, Lemma 7.5], if $\sigma = w^*\text{-lim } \overline{\Psi}_{t_n}\omega_0 \in B_0$, then $\overline{\Psi}_{t_n}\omega_0$ converges to σ in norm. Thus $\sigma = f\omega_0$, where (for a subsequence) $f(x) = \lim \overline{\Psi}_{t_n}(x)$, a.e. ($|\omega_0|$). The function f is a Borel homomorphism on a normal Borel subgroup of the support of ω_0. Identify B_0 with the set of such f and partition B_0 by $f \sim g$ if $E(f\omega_0) = E(g\omega_0)$. The number of equivalence classes is finite, for otherwise we would have a sequence $\{f_i\} \subseteq B_0$ with $E(f_i\omega_0) \neq E(f_j\omega_0)$ if $i \neq j$. By using a diagonal process, we can find an $f \in B_0$ such that (for a subsequence) $f_i\omega_0 \to f\omega_0$ in norm. But then $\|f_i\omega_0 - f_j\omega_0\| < 1/2$ for i and j large enough. Since $(f_i\omega_0)\hat{}(\alpha) \geq 1$ for all i and for all $\alpha \in E(f_i\omega_0)$, it follows that $E(f_i\omega_0) = E(f_j\omega_0)$, a contradiction.

Now choose δ as in Lemma 10 and choose $N_1 > N_0$ such that if $\alpha = \alpha_1 \cdots \alpha_n \in E(\omega_0)$ and $\alpha_i \equiv 1$ for $i \leq N_1$, then $\|\overline{\Psi}_\alpha\omega_0 - f\omega_0\| < \delta$ for some $f \in B_0$. Let $K_1 = A \times \Pi_1^{N_1}G_i$ and put $\lambda_0 = \omega_0 * m_{K_1}$. Then $\lambda_0 \neq 0$ and $\alpha \in E(\lambda_0)$ if and only if $\|\overline{\Psi}_\alpha\lambda_0 - f\lambda_0\| < \delta$ for some $f \in B_0$. We shall now identify λ_0 with its projection on the group $\Pi_{N_1+1}^\infty G_i$. One can now show that the hypotheses of Lemma 10 are satisfied (applied to λ_0 and the group $\Pi_{N_1+1}^\infty G_i$), and thus obtain functions $h_i \in L^\infty(|\lambda_0|)$ such that if $\sigma_0 = h_1\lambda_0 * \cdots * h_n\lambda_0$ then $\alpha \in E(\sigma_0)$ if and only if $\|(\overline{\Psi}_\alpha - 1)\sigma_0\| < 1/600$. Lemma 9 implies that σ_0 is a multiple of Haar measure of $\Pi_{N_1+1}^\infty G_i$. Now recall that $E(\omega_0) \subseteq E(\pi_P\omega_0)$. This implies that if $\overline{\Psi}_\alpha\omega_0 \to f\omega_0$ in norm, then $E(f\lambda_0) \subseteq E(f\pi_P\lambda_0)$, and thus $E(h_i\lambda_0) \subseteq E(h_i\pi_P\lambda_0)$ for $1 \leq i \leq n$. Therefore, $E(\sigma_0) \subseteq E(\pi_P\sigma_0)$ and hence $\pi_P\sigma_0 \neq 0$. Thus P is open in $\Pi_{N_1+1}^\infty G_i$, hence in G.

Lemma 12 now yields the existence of an M such that $\Psi_{\beta_i} \equiv 1$ for all $i > M$. Let $\theta = \beta_1 \cdots \beta_M$, let $K = \Pi_1^M G_i$ and let $\omega = w^*\text{-lim } \bar{\gamma}\overline{\Psi}_\lambda(\overline{\Psi}_\theta\mu * m_K)$ (for a subsequence). We shall identify ω with its projection on the group $A \times \Pi_{M+1}^\infty G_i$. We can repeat the argument given above beginning with the paragraph following (7) (and omitting the following two paragraphs), replacing λ by the measure ω. What was needed there was the fact that $1 \in E(\lambda)$ to ensure that $B \neq \varnothing$. We thus obtain a measure $\omega_0 = \lim \Psi_{t_n}\omega$, where $\{t_n\}$ is an F-sequence in $E(\omega)$, and functions $h_1, \ldots, h_n$ such that if $\lambda_0 = \omega_0 * m_{K_0}$ and $\sigma = h_1\lambda_0 * \cdots * h_n\lambda_0$, then $\alpha \in E(\sigma)$ if and only if $\|(\Psi_\alpha - 1)\sigma\| < 1/600$. Here $K_0 = \Pi_1^{M_0}G_i$, where $M_0 > M$, and as usual σ is identified with its projection on $A \times \Pi_{M_0+1}^\infty G_i$.

We claim now that, for some $M_1 \geq M_0$, $\sigma = \sigma * m_H$, where $H = \Pi_{M_1}^\infty G_i$. To see this, we repeat the arguments in the first part of the proof to show the existence of an integer I and a sequence $\{\alpha_n\} \subset E(\sigma)$ with $\alpha_n = \beta_1 \cdots \beta_I\gamma_n\lambda_n$, where $\{\gamma_n\lambda_n\}$ forms an F-sequence. We can also assume $d(\lambda_n) > 1$ for all n. Let $\beta = \beta_1 \cdots \beta_I$ and choose integers n_1 and n_2 such that $\gamma_{n_1}\lambda_{n_1}\gamma_{n_2}\lambda_{n_2}$ is irreducible. Now

$$\left|\int \left(\overline{\Psi}_\beta^2\bar{\gamma}_{n_1}\overline{\Psi}_{\lambda_{n_1}}\bar{\gamma}_{n_2}\overline{\Psi}_{\lambda_{n_2}} - 1\right)d\sigma\right| < \left\|\left(\overline{\Psi}_\beta\bar{\gamma}_{n_1}\overline{\Psi}_{\lambda_{n_1}} - 1\right)\sigma\right\| + \left\|\left(\overline{\Psi}_\beta\bar{\gamma}_{n_2}\overline{\Psi}_{\lambda_{n_2}} - 1\right)\sigma\right\| < \frac{2}{600}.$$

Thus, since $1 \in E(\sigma)$ and $\sigma \in I(G)$, there exists $\theta_1 \in \beta \otimes \beta$ such that $\theta_1\gamma_{n_1}\lambda_{n_1}\gamma_{n_2}\lambda_{n_2} \in E(\sigma)$. Hence, $\|(\overline{\Psi}_{\theta_1}\bar{\gamma}_{n_1}\overline{\Psi}_{\lambda_{n_1}}\overline{\Psi}_{n_2}\overline{\Psi}_{\lambda_{n_2}} - 1)\sigma\| < 1/600$, and we can repeat the argument to show that for some $\theta_2 \in \theta_1 \otimes \beta$ and some $\gamma_{n_3}\lambda_{n_3}$, $\theta_2\gamma_{n_1}\lambda_{n_1}\gamma_{n_2}\lambda_{n_2}\gamma_{n_3}\lambda_{n_3} \in E(\sigma)$. Thus we obtain a sequence $\{t_m\} \subseteq E(\sigma)$ with $t_m = \theta_m\gamma_{n_1}\lambda_{n_1}\gamma_{n_2}\lambda_{n_2} \cdots \gamma_{n_m}\lambda_{n_m}$. By repeating again the arguments in the first part of the proof, we see that $\lambda_{n_m} \equiv 1$ for m large enough, a contradiction and our claim is established.

Now let $g = w^*\text{-lim } \overline{\Psi}_{t_n}$ in $L^\infty(|\omega|)$. Then, since $C(G) \subseteq L^\infty(|\omega|)$, we have that

$$\omega_0 = g\omega = w^*\text{-lim } \bar{\gamma}\overline{\Psi}_{\lambda_n}(g\mu_0)$$

where $\mu_0 = \Psi_\theta\mu * m_K$. Since $\{\gamma_n\lambda_n\}$ is an irreducible sequence,

$$\lambda_0 = \omega_0 * m_{K_0} = w^*\text{-lim } \bar{\gamma}_n\overline{\Psi}_{\lambda_n}(g\mu_0 * m_{K_0}).$$

Also, for each i

$$h_i\lambda_0 = w^*\text{-lim } \bar{\gamma}_n\overline{\Psi}_{\lambda_n}(h_i g\mu_0 * m_{K_0}),$$

so that

$$\sigma = \prod h_i\lambda_0 = w^*\text{-lim } \bar{\gamma}_n\overline{\Psi}_{\lambda_n}\Big(\prod h_i g\mu_0 * m_{K_0}\Big).$$

Hence, by Lemma 11, $\{\lambda_n\}$ is contained in finitely many hypercosets of H which contradicts (b), and the theorem is proved.

Corollary 14. *If $G = \prod_1^\infty G_i$ and $\mu \in I(G)$ is of b.r.t. in each coordinate, then μ is a trigonometric polynomial.*

5. The general case. In this section we use Weil's characterization of compact, connected groups to prove Theorem 1. This amounts to extending our results to abritrary products and to factor groups.

Proof of Theorem 1. (a) *Arbitrary products.* Let $\mu \in I(G)$, where $G = A \times \prod_I G_\alpha$ and where, as usual, A is abelian and each G_α is a compact, connected, simple Lie group. Suppose that there exists a countable set $J \subseteq I$ and a countably infinite sequence $\{\alpha_n\} \subseteq \Gamma(\prod_J G_\alpha)$ such that $\{\alpha_n\} \subset E(\mu)$ and $\Psi_{\alpha_n|\prod_J G_\alpha}$ are all distinct. If $\nu = \mu * m_K$, where $K = \prod_{I-J} G_\alpha$, then $\nu \neq 0$, $\nu^* \in I(A \times \prod_J G_\alpha)$, and $\{\alpha_n\} \subseteq E(\nu^*)$. If ν is of b.r.t. in each coordinate, then Theorem 13 implies that $\Psi_{\alpha_n|\prod_J G_\alpha}$ are not all distinct, a contradiction. Thus we can assume that μ is not of b.r.t. for all but a finite number of coordinates. Let $H = A \times \prod_I H_\alpha$ where $H_\alpha = Z_\alpha$ if μ is not of b.r.t. in the αth coordinate and $H_\alpha = G_\alpha$ otherwise. We claim that $\sigma = \mu_{|H} \neq 0$ and that $\sigma \in I(H)$. This will complete the proof since we can then apply Theorem 6 to σ and use the fact that $\pi_K\sigma = \pi_K\mu$ for K a normal subgroup of H.

If $|\mu|(H) = 0$ then the regularity of μ implies the existence of open sets $V_n \supset H$ such that $|\mu|(V_n) < 1/n$. Hence, $|\mu|(\bigcap_1^\infty V_n) = 0$, and since each V_n restricts only a finite number of coordinates, it follows that $|\mu|(A \times \prod_J Z_i \times \prod_{I-J} G_\alpha) = 0$ where J is countable. However, the results in the first two paragraphs of §4 show that this is impossible since we are assuming that μ is of b.r,t. for only finitely many coordinates.

To show that $\sigma \in I(H)$ let $\gamma \in \Gamma(H)$, let $\varepsilon > 0$, and let V be an open set containing H such that $|\mu|(V - H) < \varepsilon$. If $\beta \in \Gamma(G)$ is chosen such that $\Psi_{\beta|H} = \gamma$, then

$$\int \bar{\gamma}\, d\sigma = \int \overline{\Psi}_\beta\, d\mu - \int \overline{\Psi}_\beta\, d\mu|_{G-V} - \int \Psi_\beta\, d\mu|_{V-H}.$$

The last integral on the right has absolute value less than ε. Since V restricts only a finite number of coordinates, β can be chosen (with $d(\beta)$ large) so as to make the second integral less than ε (see the third paragraph in §4). Since $\mu \in I(G)$, it follows that $\sigma \in I(H)$.

(b) *Factor groups*. If G is a group for which Theorem 1 is valid and H is a closed, normal subgroup of G, then Theorem 1 is valid for G/H. Let $\sigma \in I(G/H)$ and choose $\mu \in I(G)$ such that $\mu^* = \sigma$ and $\mu * m_H = \mu$. There exists a closed, normal subgroup K of G such that $0 \neq \pi_K\mu = \nu * \eta$ where ν is invertible and η is K-canonical. Now

$$0 \neq \pi_K\mu = \pi_K(\mu * m_H) = \pi_K\mu * m_H.$$

This implies that K is open in HK and thus $\pi_K\mu = \pi_{HK}\mu$. By Lemma 3 we have the equality

$$\pi_{HK/H}\sigma = (\pi_K\mu)^* = (\nu * \eta)^* = \nu^* * \eta^*.$$

By comparing transforms it is easy to see that ν^* is invertible in $M(G/H)$ and that $\eta^* \ll m_{HK/H}$.

6. A counterexample. Theorem 1 fails for the disconnected case. Let G be the semidirect product $T \times T \times | Z_2$, where T is the circle and Z_2 the group of order 2. Z_2 acts on $T \times T$ by $\alpha(t_1, t_2) = (t_2, t_1)$ for $\alpha \neq e$. Let μ_1 be Haar measure on $T \times e \times e$, μ_2 Haar measure on $e \times T \times e$ and μ_3 Haar measure on $T \times T \times e$. If $\mu = \mu_1 + \mu_2 - \mu_3$ then μ is a central idempotent. Suppose there exists a normal subgroup H of G for which $\pi_H\mu = \nu * \eta \neq 0$. It is easily seen that the only normal subgroup G for which $\pi_H\mu \neq 0$ is $T \times T \times e$ which has finite index in G. Thus $\mu = \pi_H\mu = \nu * \eta = \eta$ since η and μ are both idempotents. However, this implies that $\mu \ll m_{T\times T\times e}$ which is not true.

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